Discrete Random Variables

Chapter 3 – Lecture 9

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Expected Value

- The most important characteristic of a random variable is its **expected value**. Let's start with an easy example.
- How do we find the average of the numbers $1, 2, \ldots, 6$? You learned before kindergarten to simply *add them and divide by 6*!
- Well, after you graduated with a kindergarten degree, you learned that this is equivalent to:

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6}.$$

- This is assuming that each value "appears" with equal chance.
- In other words, if we let X be a random variable that takes on values $\{1, 2, \ldots, 6\}$ with equal chance, the average then is simply the *weighted means* of all values in the range of X.

Definition 1 (Expected Value).

The **expected value** of the discrete random variable X having I as its set of possible values is defined by

$$E(X) = \sum_{x \in I} x\, P(X = x).$$

- For instance, let's consider the experiment of rolling a fair die twice and let the random variable X denote the *smallest* of the two numbers rolled.
- Then the expected value of X is

$$E(X) = 1 \times \frac{11}{36} + 2 \times \frac{9}{36} + 3 \times \frac{7}{36} + 4 \times \frac{5}{36} + 5 \times \frac{3}{36} + 6 \times \frac{1}{36} = 2.528.$$

- The expected value is also known as *expectation*, or *mean*, or *first moment* and *average value* of X.

Example 1.

Consider Example 1 in Lecture 8 again. (Grabbing two coins from a bag containing three dimes and two quarters.) What is the expected value of the amount you grabbed from your pocket?

- We already have the probability mass function:

$$P(X = 20) = \frac{3}{10},\ P(X = 35) = \frac{3}{5},\ P(X = 50) = \frac{1}{10}$$

- Hence the expected value is

$$E(X) = 20 \times \frac{3}{10} + 35 \times \frac{3}{5} + 50 \times \frac{1}{10} = 32 \text{ cents.}$$

Example 2.

Joe and his friend make a guess every week whether the Dow Jones index will have risen at the end of the week or not. Both put \$10 in the pot. Joe observes that his friend is just guessing and is making his choice by the toss of a fair coin. Joe asks his friend if he could contribute \$20 to the pot and submit his guess together with that of his brother. The friend agrees. In each week, however, Joe's brother submits a prediction opposite to that of Joe. The person having a correct prediction wins the entire pot. If more than one person has a correct prediction, the pot is split evenly. How favorable is the game to Joe and his brother?

- Let X denote the payoff to Joe and his brother in any given week.
- Exactly one of Joe and his brother will have a correct prediction every week. If the friend is wrong he wins nothing, and if he is correct, he shares the \$30 with either Joe or his brother.
- Hence the expected payoff is

$$E(X) = \frac{1}{2} \times 30 + \frac{1}{2} \times 15 = 22.5 \text{ dollars.}$$

Example 3.

Three friends go to the cinema together every week. Each week, in order to decide which friend will pay for the other two, they all toss a fair coin into the air simultaneously. They continue to toss coins until one of the three gets a different outcome from the other two. What is the expected value of the number of trials required?

- Let X denote the number of trials until one of the three friends gets a different outcome from the other two.
- The probability that any given trial lead to three equal outcome is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. Thus,

$$P(X = j) = \left(\frac{1}{4}\right)^{j-1} \left(\frac{3}{4}\right) \text{ for } j = 1, 2, \ldots$$

- Hence, the expected value of X is

$$E(X) = \sum_{j=1}^{\infty} j \left(\frac{1}{4}\right)^{j-1} \left(\frac{3}{4}\right) = \frac{3/4}{(1 - 1/4)^2} = \frac{4}{3}.$$

Sum of Infinite Series

- The evaluation of the last infinite series is made possible by differentiating the following term by term

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{j=0}^{\infty} x^j, \qquad \text{for all } |x| < 1$$

- After differentiation (assuming we can differentiate) we have,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{j=1}^{\infty} jx^{j-1}$$

- Raymond Yan, a prodigy from 11B, came up with a brilliant method:

$$\sum_{j=1}^{\infty} jx^{j-1} = \sum_{j=1}^{\infty} x^{j-1} + \sum_{j=2}^{\infty} x^{j-1} + \cdots = \frac{1}{1-x} + \frac{x}{1-x} + \cdots = \frac{1}{(1-x)^2}$$

The Law of Large Numbers Revisited

- Previously we stated the *Law of Large Numbers* intuitively as:
- *As the number of trials of a chance experiment increase without bound, the relative frequency of an event will converge to its theoretical probability.*
- Let X be a random variable associated to a single rolling of a fair die. Then it is easy to see that

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3\frac{1}{2}.$$

- By the law of large numbers, the relative frequency of rolling a j is closely equal to its theoretical probability $\frac{1}{6}$ for every $j = 1, ..., 6$.
- From here it follows that the *average number* per roll is is close to $\frac{1}{6}(1 + 2 + \cdots + 6) = 3\frac{1}{2} = E(X)$.

The Law of Large Numbers Revisited

- Now let X be a random variable that is defined on the probability space of a chance experiment.
- Let X_k be the value of the random variable X in the kth repetition of the game. (X_k itself is a random variable having the same range and distribution as X. This is often referred to as **i.i.d.** by statisticians, and is short for **independently identically distributed**.)
- Then the **Theoretical Law of Large Numbers for Expected Value** can be stated as
- *The average value $\frac{1}{n}(X_1 + X_2 + \cdots + X_n)$ over the first n repetitions of the game will converge with probability 1 to a constant as $n \to \infty$ and this constant is equal to the expected value $E(X)$.*
- Thus, we may view expected value of a random variable as a **long-term average**. This is justified by the law of large numbers!

Expected Value and Risk

- In measuring the payoffs of a casino game, or a stock, the expected value $E(X)$ is very informative, but usually not sufficient when X is the random payoff in a **nonrepeatable** game. (Why?)
- Suppose your investment has yielded a profit of \$3,000 and you must choose between the following two options:
  1. Take the sure profit of \$3,000.
  2. Reinvest the profit of \$3,000 under the scenario that this profit increases to \$4,000 with probability 0.8 and is lost with probability 0.2.
- The expected profit of the second option is $0.8 \times \$4,000 + 0.2 \times \$0 = \$3,200$, and is larger than \$3,000.
- Nevertheless, most people would prefer the first option!
- The downside to the second option is its **risk**. The risk of a game or stock is measured by its variance.

Example 4 (The best-choice problem).

Your friend proposes the following wager: twenty people are requested, independently of one another, to write a number on a piece of paper (the papers should be evenly sized). They may write any number they like, no matter how high. You fold up the twenty pieces of paper and place them randomly onto a tabletop. Your friend opens the papers one by one. Each time he opens one, he must decide whether to stop with that one or go on to open another one. Your friend's task is to single out the paper displaying the highest number. Once a paper is opened, your friend cannot go back to any of the previously opened papers. He pays you one dollar if he does not identify the paper with the highest number on it, otherwise you pay him five dollars. Do you take the wager? If your answer is no, what would you say to a similar wager where 100 people were asked to write a number on a piece of paper and the stakes were one dollar from your friend for an incorrect guess against ten dollars from you if he guesses correctly?