Introduction to Probability

Lecture 1

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Ancient History

- Who is greater in the preindustrial economy: inventor of the wheel or the crafter of the first pair of dice?
- Did you know that a "dice"-like object existed as early as 3500 B.C.?
- By the end of the 17th century, the Dutch astronomer Christiaan Huygens (1629-1695) laid the foundation for probability theory.
- Huygens originally introduced the concept of the expected value, which plays a vital role in probability theory.
- In 1713, the Swiss mathematician Jakob Bernoulli (1654-1705) presented the first general theory for calculating probabilities.
- In 1812, the great French mathematician Pierre Simon Laplace (1749-1827) made the greatest contribution in the history of probability theory.

Modern History

- Although probability theory was initially a product of questions posed by gamblers, it in its modern form plays a great role in many fields.
- Better judicial and medical decisions result from an elementary knowledge of probability theory.
- It is essential in the field of finance (actuarial science) and economics.
- The stock market, "the largest casino in the world," cannot do without probability.
- The telephone network with its randomly fluctuating load could not have been economically designed without the aid of probability theory.
- Call-centers and airline companies apply probability theory to determine how many telephone lines and service desks will be needed based on expected demand.
- Also, astronomers use probability theory extensively.

Modern History

- Laplace was right when he wrote almost 200 years ago in his Théorie Analytique des Probabilitiés:

  > *The theory of probabilities is at bottom nothing but common sense reduced to calculus; it enables us to appreciate with exactness that which accurate minds feel with a sort of instinct for which ofttimes they are unable to account... . It teaches us to avoid the illusions which often mislead us; ... there is no science more worthy of our contemplations nor a more useful one for admission to our system of public education.*

- So let's begin our exciting journey to the world of chance!

Probability and Simulation

- Many people have difficulties in developing an intuition for probability because probability is often counter-intuitive.
- In no other branch of mathematics is it so easy to make mistakes as in probability theory!
- Did you know that the genius Leibniz had difficulties in calculating the probability of throwing 11 with one throw of two dice?
- Probability is also similar to the concept of limit where it is extremely hard to define via a formal mathematical definition.
- Computers offer a great help in developing an intuition for probability.
- With computer simulation, a concrete probability situation can be imitated on the computer.
- Computer simulation is also a practical tool for tackling probability problems that are too complicated for scientific solution.

Outline of the Course

- One of the main goals of this course is to introduce two pillars of probability theory:
  - the Law of Large Numbers
  - The percentage of tosses of heads on a fair coin can be as close to 0.5 as you can imagine provided that the number of tosses is large enough.
  - the Central Limit Theorem (CLT)
  - How often the coin must be tossed in order to reach a prespecified precision for the percentage can be identified with the CLT.
- With the help of the CLT, we will also study the idea of the confidence interval. (How many simulations is enough?)
- IF we have time, we can go into more interesting topics like random walk, Brownion motion, and the Black-Scholes formula for the pricing of options, an important application in finance.

The Gambler's Fallacy

- In the midst of a coin-tossing game, after seeing 10 tails in a row, will the chance of tossing heads in the next toss be larger?
- Or, if we have rolled a fair die and got no six's in 100 tries, we are sure that finally we will roll a six on the 101st try?
- These notions are known as the *gambler's fallacy*.
- Only in the *long run* are we expected to see the ratio of the numbers of head and tails approach 1 by the law of large numbers.
- Does the absolute difference of number of heads and tails approach 0?
- No! Actually, quite often, if the number of heads initially exceeds the number of tails, it will STAY ahead for a long time before suddenly reverting to the other direction!
- This absolute difference, as we will see later, is an example of a random walk. You must feel very excited right now!

Relative Frequency

- Suppose the weather statistic shows that in the past 200 years, on average, it rained 7 of 30 days in June, with no definite pattern on which days.
- Assuming things do not change, then you can say that probability of raining on June 15, 2016 is approximately _____. Why?

Definition 1 (Relative Frequency).

The relative frequency of the event A in n repetitions of the experiment is defined as

$$f_n(A) = \frac{n(A)}{n}$$

where $n(A)$ is the number of times that event A occurred in the n repetitions of the experiment.

Emperical Law of Large Numbers

Emperical Law of Large Numbers

The relative frequency with which event A occurs will fluctuate less and less as time goes on, and will approach a limiting value as the number of repetitions increases without bound.

- Intuitively, we would like to define the probability of the occurrence of the event A in a single repetition of the experiment as the limiting number to which the relative frequency $f_n(A)$ **converges** as n increases. But...
- A formal approach requires the notion of the *sample space*, with a *probability measure* that satisfies three axioms.
- This formal approach is precisely based on the properties of the relative frequency.
- This leads to the *theoretical law of large numbers*.

Theoretical Law of Large Numbers

- Let's consider the experiment of tossing a fair coin nonstop.
- Any outcome of this experiment has the same form as $\omega = (H, T, T, H, H, T, T, T, ...)$ and we define

$$K_n(\omega) = \text{the number of heads in the first } n \text{ tosses}$$

- Intuitively, we expect that "nature" will guarantee that ω will satisfy

$$\lim_{n\to\infty} \frac{K_n(\omega)}{n} = \frac{1}{2}$$

- There are many conceivable sequences ω for which this does not converge to $1/2$. Nevertheless, "nature" (Jade Emperor) dictates that the set of such outcomes is "negligibly small." How remarkable!
- Please respect the Jade Emperor.

Theoretical Law of Large Numbers

- The formulation of the *theoretical law of large numbers* requires advanced concepts, but in words it can be stated as

  > *If a certain chance experiment is repeated independently an unlimited number of times under identical conditions, then the relative frequency of A will indeed converge to P(A), the theoretical probability of A occuring in a* single *repetition of the experiment.*

- To give you a taste, the formal version is,

$$P\left(\{\omega : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k(\omega) = \mu\}\right) = 1,$$

and the proof requires the use of moment generating functions, which is covered in G12 options topic. Can't wait for G12, guys.

Birthday Problem – Simulation Approach

Example 2 (Birthday Problem).

What is the probability that, in a group of randomly chosen people, at least two of them will have been born on the same day of the year? How many people are needed to ensure a probability > 0.5?

- Using Python we can randomly generate a list of birthdays using the "random" module. (We can also use Maple, Matlab, etc.)
- We test whether a list of birthdays contains duplicates by calling the "count" function.
- We run this trials many times using the for-loop and count the total number of lists that contains duplicates.
- Finally we divide the number trials by the number of lists that contains duplicates. This gives us the relative frequency.
- By the law of large numbers, this should converge to the theoretical probability which we need to find!

Python Code

```
import random
classSize = 25
numTrials = 10
dupeCount = 0
bdayList = []
for trial in range(numTrials):
    bdayList = []
    for i in range(classSize):
        newBDay = random.randrange(365)
        bdayList.append(newBDay)
    foundDupe = False
    for num in bdayList:
        if bdayList.count(num) > 1:
            foundDupe = True
    if foundDupe == True:
        dupeCount = dupeCount + 1
prob = dupeCount / numTrials
print("The probability of shared birthday is ", prob)
```

Birthday Problem – Theoretical approach

- The probability can be calculated as

$$P_n = 1 - \frac{365 \times 364 \times \cdots \times (365 - n + 1)}{365^n}$$

- The following table plots some values

n	15	20	25	30	40	50	75
P_n	0.2529	0.4114	0.5687	0.7063	0.8912	0.9704	0.9997

- In fact, we have

$$P_n \approx 1 - e^{-\frac{1}{2}n(n-1)/365}$$

- This can be derived using a linearization. (See homework.)
- There are many interesting probability problems that defy our intuition. Since time is limited, we'll come back to them later.
- Next we will be studying the axiomatic approach to probability.

Homework Problem

Homework (Due next class)

The Birthday Problem can be generalized into this problem. Suppose you randomly and independently drop n balls in c compartments such that $c >> n$. Show that the probability P_n of having at least two balls in the same compartment is

$$P_n = 1 - \frac{c \times (c-1) \times \cdots (c-n+1)}{c^n} \approx 1 - e^{-\frac{1}{2}n(n-1)/c}$$

- *Hint:* use the linearization $e^{-x} \approx 1 - x$ for x sufficiently close to 0.