Chinese name: ____________ English name: ____________ Homeroom: ______ Math Class: ______

The Chain Rule

1. In class, we showed that if $F(x,y)$ is differentiable and $F(x,y)=0$ defines y implicitly as a differentiable function of x, then at any point $F_y \neq 0$, $\mathrm{d}y/\mathrm{d}x = -F_x/F_y$. Now generalize this. Suppose that the equation $F(x,y,z)=0$ defines the variable z implicitly as a function $z=f(x,y)$. Assuming that F and f are differentiable functions, use the Chain Rule to deduce that
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

2. Use result from Question 1 to find $\partial z/\partial x$ and $\partial z/\partial y$ at (π,π,π) if $\sin(x+y)+\sin(y+z)+\sin(x+z)=0$.

3. Assume that $w=f(s^3+t^2)$ and $f'(x)=\mathrm{e}^x$. Find $\partial w/\partial t$ and $\partial w/\partial s$.

4. Assume that $w=f(ts^2, s/t)$, $f_x(x,y)=xy$ and $f_y(x,y)=x^2/2$. Find $\partial w/\partial t$ and $\partial w/\partial s$.

5. Let $T=f(x,y)$ be the temperature at the point (x,y) on the circle $x=\cos t$, $y=\sin t$, $0\le t\le 2\pi$ and suppose that
$$\frac{\partial T}{\partial x} = 8x-4y, \quad \frac{\partial T}{\partial y} = 8y-4x.$$
   (a) Find where the maximum and minimum temperatures on the circle occur by examining the derivatives $\mathrm{d}T/\mathrm{d}t$ and $\mathrm{d}^2T/\mathrm{d}t^2$.
   (b) Suppose that $T=4x^2-4xy+4y^2$. Find the maximum and minimum values of T on the circle.

6. Under mild continuity restrictions, it is true that
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b g(t,x)\,\mathrm{d}t = \int_a^b \frac{\partial}{\partial x} g(t,x)\,\mathrm{d}t.$$
Using this fact and the Chain Rule, find the derivative of
$$F(x) = \int_0^{x^2} \sqrt{t^4+x^3}\,\mathrm{d}t$$
with respect to x by first writing
$$F(x) = \int_a^b g(t,x)\,\mathrm{d}t = G(u,x) = \int_0^u g(t,x)\,\mathrm{d}t, \quad \text{where } u=f(x).$$

Gradient and Directional Derivatives

7. Find the gradient of $f(x,y)=xy^2$ at the point $(2,-1)$. Then sketch the gradient together with the level curve that passes through the point.

8. Find the directional derivative of the function $f(x,y,z)=xy+yz+zx$ in the direction $\mathbf{u}=3\mathbf{i}+6\mathbf{j}-2\mathbf{k}$ at the point $P_0(1,-1,2)$.

9. Let $f(x,y)=x^2-xy+y^2-y$. Find the direction $\mathbf{u}$ and the values of $D_{\mathbf{u}}f(1,-1)$ for which
   (a) $D_{\mathbf{u}}f(1,-1)$ is largest
   (b) $D_{\mathbf{u}}f(1,-1)$ is smallest
   (c) $D_{\mathbf{u}}f(1,-1)=0$
   (d) $D_{\mathbf{u}}f(1,-1)=4$
   (e) $D_{\mathbf{u}}f(1,-1)=-3$

10. The derivative of $f(x,y,z)$ at a point P is greatest in the direction of $\mathbf{v}=\mathbf{i}+\mathbf{j}-\mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$.

(a) What is ∇f at P? Give reasons for your answer.

(b) What is the derivative of f at P in the direction of $\mathbf{i}+\mathbf{j}$?

11. Sketch $x^2 - xy + y^2 = 7$ together with ∇f and the tangent line at the point $(-1, 2)$. Then write an equation for the tangent line.

12. Given a constant k and the gradients

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \quad \nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k},$$

establish the algebra rules for gradients (Slide 22, Lecture 3).

13. Find the equation of the tangent plane and the normal line at the point $P_0(2, 03, 18)$ of the surface $x^2 + y^2 - 2xy - x + 3y - z = -4$.

14. Find the parametric equations for the line tangent to the curve of intersection of surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$. (Hint: Consider two surfaces as level surfaces. The tangent line must be orthogonal to both gradient vectors.)

15. Consider the function $y = f(x)$. The **differential** $\mathrm{d}y$ obtained from moving from x_0 to $x_0 + \mathrm{d}x$ is

$$\mathrm{d}y = f'(x_0)\,\mathrm{d}x$$

which can be used to estimate the change of y after a change in x by $\mathrm{d}x$.

Now consider the function $z = f(x, y)$. If we move from $P_0(x_0, y_0)$ to $P_1(x_0 + \mathrm{d}x, y_0 + \mathrm{d}y)$, the resulting change Δz can be estimated by the **total differential**

$$\mathrm{d}f = \left.\frac{\partial f}{\partial x}\right|_{P_0} \mathrm{d}x + \left.\frac{\partial f}{\partial y}\right|_{P_0} \mathrm{d}y. \tag{1}$$

Next week, we will discuss this in detail. Now I try to give you an intuition. Recall that the directional derivative of f at P_0 in the direction of $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is

$$\left(\frac{\mathrm{d}f}{\mathrm{d}s}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{2}$$

Notice that there is also a $\mathrm{d}f$ here. We can define $\mathrm{d}s$ as the **arc length differential** to be a change in distance moving away from the point P_0 along some curve $\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j}$. We can define

$$x(s) = x_0 + su_1$$
$$y(s) = y_0 + su_2$$

Since $\mathrm{d}x/\mathrm{d}s = u_1$ and $\mathrm{d}y/\mathrm{d}s = u_2$, equation (2) can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} \tag{3}$$

Does equation (3) look familiar to you? How is this related to equation (1)? Meow Meow Meow?